INTEGRATING GRAPH-THEORETIC METHODS INTO TRANSPORTATION LOGISTICS: A COST-MINIMIZATION APPROACH

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Abstract:

The transportation problem remains a central concern in logistics and operations research, aimed at determining the most efficient way to move goods from sources to destinations while minimizing cost. This study explores the integration of graph-theoretic methods into transportation logistics, providing a robust mathematical framework for analyzing and solving complex transport networks. By modeling transportation systems as weighted graphs and applying tools such as shortest path algorithms, minimum spanning trees, and minimum-cost flow models, this approach offers deeper insights into cost minimization, network resilience, and operational efficiency. Additionally, the application of regularization techniques enhances solution stability and robustness. The study concludes that graph-based frameworks are not only mathematically sound but also practically advantageous for modern logistics planning.

Keywords: Graph Theory, Transportation Optimization, Network Flow Analysis, Spanning Subgraph.

Introduction

The transportation problem, a cornerstone of operations research and logistics, centers on determining the most cost-effective method for moving goods or resources from multiple supply points to various demand destinations. The objective is to satisfy demand and supply constraints while minimizing overall transportation costs. Traditional optimization techniques have addressed this problem effectively; however, the integration of graph-theoretic methods offers a more robust, flexible, and visually intuitive framework for addressing complex logistics challenges.

In graph-theoretic terms, transportation systems are modeled as networks, where nodes represent locations (such as warehouses, factories, or markets) and edges denote the possible routes connecting these points, often weighted by cost, distance, or time. This structure enables the application of powerful algorithms from graph theory to identify optimal paths, minimize routing

Nigerian Journal of Logistics and Transport 2025,Vol. 17 (1) DOI: 10.61955/VLJZXI

costs, and evaluate network efficiency. Techniques such as shortest path algorithms, minimum spanning trees, and minimum-cost network flows are central to this approach, enabling efficient computation of optimal transportation strategies

Beyond optimization, graph-theoretic methods contribute to transportation network design by enabling the analysis of resilience and vulnerability. Concepts such as dominating sets, edge domination, and small-world or scale-free network properties help identify critical infrastructure, potential bottlenecks, and strategic points for capacity expansion, such insights are crucial in logistics, where disruptions can have cascading effects on cost and efficiency. By integrating graph-theoretic principles into transportation logistics, organizations can not only optimize cost but also gain deeper insight into network structure and performance.

Literature Review

The integration of graph theory into transportation and logistics management has revolutionized how researchers and practitioners analyze, optimize, and predict the behavior of complex networks. Graph-based models allow transportation systems to be represented as structured networks consisting of nodes (warehouses, factories, markets, depots, stations, or cities) and edges (roads, railways, or flight paths). This mathematical abstraction enables the use of graph algorithms to solve a wide range of logistical and operational problems. It simplifies complex systems into manageable computational frameworks that can be optimized for efficiency, reliability, and cost reduction. According to Banasiak and Falkiewicz (2015), graph-based transportation modeling offers enhanced visual and analytical clarity, which aids in identifying the most efficient transport strategies under various constraints such as time, distance, or capacity.

One of the foundational contributions of graph theory to logistics lies in shortest-path and minimum-cost flow algorithms. These techniques, such as Dijkstra's and Bellman-Ford algorithms, are applied to determine the most cost-efficient or time-efficient paths in transportation networks. As highlighted by Bharathi and Selvi (2017) and Asano (2022), such methods have been particularly instrumental in reducing operational costs and improving delivery times in supply chain systems. Additionally, minimum spanning tree algorithms are used for network design, ensuring all nodes are connected with minimal total edge weight. Vaidyanathan and Ahuja (2010) emphasized that integrating cost, capacity, and flow constraints into these models allows for solving more realistic transportation problems. Entropic and

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quadratic regularization techniques (Haasler et al., 2021) further enhance these solutions by improving numerical stability and ensuring convergence in complex optimization environments.

Graph-based models have also facilitated a deeper understanding of the structural properties of transportation systems. Network resilience, connectivity, and centrality measures derived from graph theory enable analysts to assess the vulnerability and robustness of transport infrastructures. Derrible and Kennedy (2011) explored how transportation networks can exhibit small-world and scale-free properties, revealing patterns of redundancy and criticality that influence network performance. Similarly, Guze (2019) applied edge domination and dominating set concepts to identify critical links within logistics systems, helping prioritize maintenance and expansion. These approaches make it possible to simulate network disruptions, analyze their cascading effects, and propose strategic interventions for minimizing system downtime and inefficiency.

Recent developments extend graph methods into intelligent transportation systems (ITS), where real-time data from sensors and GPS devices are integrated into graph-based models for dynamic routing and congestion management. Machine learning-enhanced graph techniques; such as Graph Neural Networks (GNNs), are being applied to predict traffic flow, optimize delivery routes, and detect bottlenecks in large-scale logistics networks. These innovations demonstrate the growing synergy between graph theory, optimization, and artificial intelligence in modern logistics. According to Essid and Solomon (2017), the adaptability of graph models to real-time data makes them particularly effective in dynamic environments, where decision-making must be both rapid and data-driven. The need for a robust, flexible, and visually intuitive framework that goes beyond traditional optimization to provide deeper insights into cost minimization, network resilience, and operational efficiency, specifically by integrating regularization techniques to enhance solution stability and robustness, and providing analysis on resilience and vulnerability (e.g., critical infrastructure and bottle necks).

Preliminaries

We introduce key concepts, notations, and definitions essential to the development of graph-theoretic models for transportation logistics.

Graph: A graph is a pair $G = (V, E)$, where V is a set of vertices (nodes) and $E \subseteq V \times V$ is a set of edges (links). If the edges are ordered pairs, the graph is said

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to be directed; otherwise, it is undirected. [Talbot & Welsh, D. (2006)]

Weighted Graph : A weighted graph is a graph G = (V, E) along with a weight function w : E → $R_{\geq 0}$, assigning a non-negative cost w(e) to each edge e ∈ E. [Frank Harary (1959)]

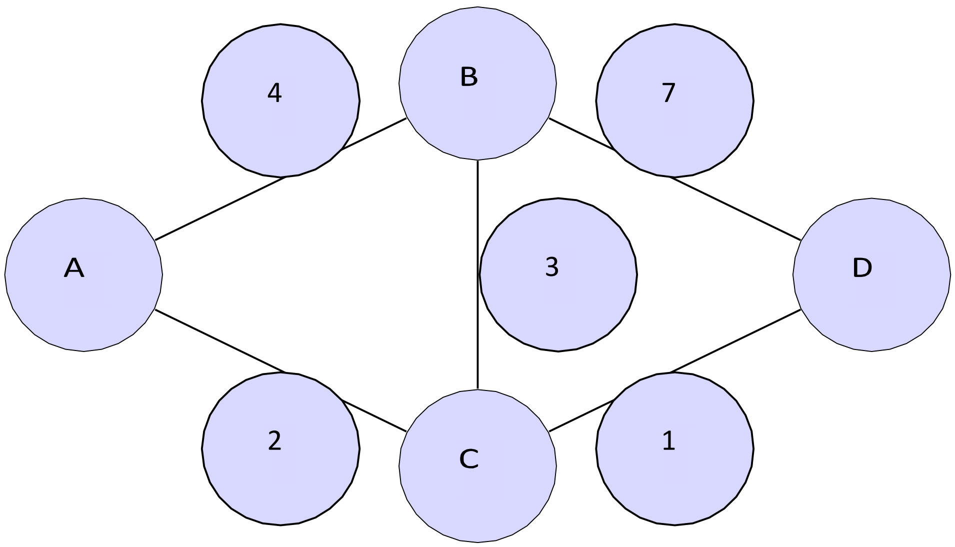

Figure 1: A weighted undirected graph $G = (V, E)$ with vertex set $\{A, B, C, D\}$ and non-negative edge weights.

Flow Network: A flow network is a directed graph G = (V, E) with a capacity function c: E → $R_{\geq 0}$, a designated source node s ∈ V, and a sink node t ∈ V. A flow is a function f: E → $R_{\geq 0}$ satisfying:

i. (Capacity constraint): $0 \leq f(e) \leq c(e)$ for all e ∈ E,

ii. (Flow conservation): For all $v \in V \setminus \{s, t\}$, $\sum_{e \in \delta^- v} f(e) = \sum_{e \in \delta^+ v} f(e)$, where $\delta^-(v)$ and $\delta^+(v)$ are the sets of incoming and outgoing edges at v, respectively.

Feasible Flow in a Transportation Network: Given a transportation problem with supplies a_i at source nodes and demands b_j at destination nodes, a feasible flow on a graph G = (V, E) is a function f: E → $R_{\geq 0}$ that satisfies:

i. Flow conservation at intermediate nodes,

ii. Satisfaction of supply/demand at source and sink nodes,

iii. Non-negativity and capacity constraints,

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iv. The condition $\sum_i a_i = \sum_i b_i$

This is according to work of Nandhini M & Ramya M (2022)

Minimum-Cost Flow in a Network: Given a flow network with edge costs w: E → $R_{\geq 0}$, a feasible flow f is said to be a minimum-cost flow if it minimizes the total cost:

$$C(f) = \sum_{e \in E} \mathrm{w}(\mathrm{e}) \cdot \mathrm{f}(\mathrm{e}) \qquad (1)$$

over all feasible flows. (Vaidyanathan & Ahuja, (2010))

Entropically Regularized Cost Function: For a feasible flow f, the entropically regularized cost function

is given by:

$$C_\epsilon(f) = \sum_{e \in E} \mathrm{f}(\mathrm{e}) \cdot \mathrm{w}(\mathrm{e}) + \varepsilon \sum_{e \in E} \mathrm{f}(\mathrm{e}) \log \mathrm{f}(\mathrm{e}) \qquad (2)$$

where ε > 0 is a regularization parameter. This function promotes uniqueness and numerical stability in optimization. (Tenetov et al , (2018)

Edge Dominating Set in a Network: A subset D ⊆ E is said to be an edge dominating set if for every vertex v ∈ V, there exists at least one edge e ∈ D such that v ∈ e (i.e., v is incident to an edge in D). This concept is useful in identifying critical routes and maintaining network reachability under cost constraints.

Methodology

This study adopts a graph-theoretic framework to model, analyse, and optimize transportation logistics networks, focusing on cost minimization and structural efficiency. The methodology is structured around three core strategies derived from graph theory: spanning subgraph optimization, regularized flow cost minimization, and dominating edge set reduction. Each technique corresponds to a formal theorem presented and justified within this framework.

Graph Modelling of Transportation Networks

Transportation systems are modelled as *graphs* $G = (V, E, w)$, where: V represents locations (e.g., warehouses, distribution centres, or terminals), E denotes transportation routes (roads, tracks, or flight paths), $w(e) \geq 0$ assigns a cost (monetary, time, or distance) to each edge $e \in E$.

Depending on the structure of the real-world system, the graph may be undirected (for symmetric routes) or directed (for one-way or constrained flows).

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Spanning Subgraph Optimization

To identify the most cost-effective backbone of the transportation network, we apply a minimum-cost spanning subgraph method. Using a connected, undirected, weighted graph, we demonstrate that if a feasible flow exists across the full graph, then a connected spanning subgraph $T \subseteq G$ can be constructed that: Preserves the feasibility of the flow (i.e., satisfies supply and demand constraints), Minimizes the total transportation cost. This is formalized in equation 1, and operationally implemented using greedy algorithms such as Kruskal's or Prim's algorithm, adapted to flow feasibility constraints.

Regularized Flow Cost Minimization

To ensure solution stability and handle degenerate or sparse transport networks, we employ **entropic regularization**. We define a regularized cost functional:

$$C_\epsilon(f) = \sum_{e \in E} f(e) \cdot w(e) + \varepsilon \sum_{e \in E} f(e) \log f(e) \qquad (3)$$

where $\varepsilon > 0$ introduces a smooth convex penalty that encourages numerical stability and uniqueness. This regularized optimization is shown in equation 2 to always admit a unique optimal flow, solvable using Sinkhorn-type algorithms or iterative scaling methods.

Flow Reduction via Dominating Edge Sets

Recognizing that full edge utilization may not be efficient or necessary, we explore flow reduction techniques using **dominating edge sets**. As shown in equation 3, if each node is incident to at least one edge from a selected subset $D \subseteq E$, then a new flow f_D supported only on D can be constructed such that:

i. f_D remains feasible,

ii. The total cost is no greater than the original flow f.

This approach simplifies the network while preserving operational feasibility, reducing complexity and computational cost in real-time logistics planning.

Visualization and Validation

Each theoretical result is supplemented by graphical illustrations. These visualizations clarify' s the role of dominating edges, feasible sub flows, and the structure of optimal subgraphs. Validation of the methodology is conducted through synthetic examples, ensuring the theorems hold under practical assumptions.

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Main Results

Theorem 1. Let $G = (V, E, w)$ be a connected, weighted, undirected graph representing a transportation network, where each edge weight $w(e)$ corresponds to the cost of transporting goods. If there exists a feasible flow on G satisfying all supply and demand constraints, then there exists a spanning subgraph $T \subseteq G$ that is connected and realizes the minimal possible transportation cost under the same feasibility conditions.

Proof. Assume the graph G satisfies the condition that total supply equals total demand, i.e., $\sum_i a_i = \sum_i b_i$ where a_i and b_j represent supply and demand at respective vertices. Let a feasible flow f be computed on G using a standard minimum-cost flow algorithm such as the Successive Shortest Path algorithm. Define the subgraph $T \subseteq G$ such that an edge $e \in T$ if and only if $f(e) > 0$. Since f satisfies all flow conservation and capacity constraints, and all demand and supply nodes are covered, the subgraph T spans all relevant vertices involved in the transportation process. Furthermore, the total cost incurred under this subgraph is given by;

$$C(f) = \sum_{e \in E} w(e) \cdot f(e) \qquad (4)$$

which is minimal due to the optimality of the flow algorithm applied. Thus, T is a cost-minimizing, connected, and feasible subnetwork.

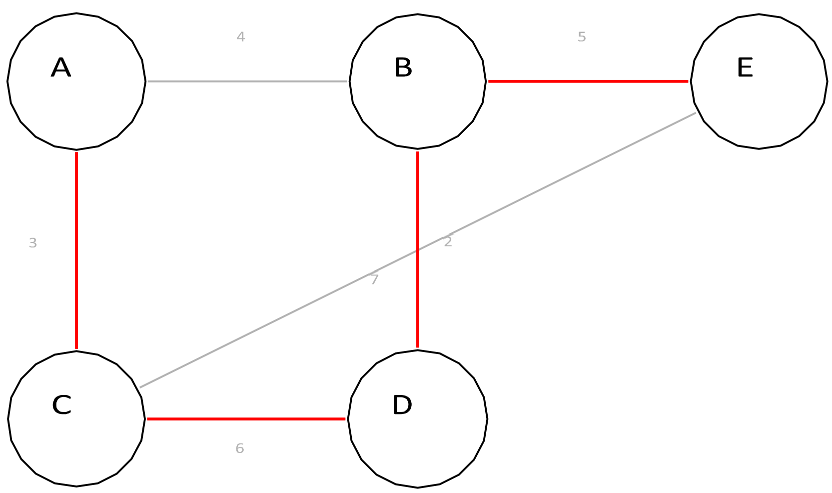

Figure 2: A connected, weighted, undirected graph G with edge weights (gray), and a minimum-cost spanning subgraph T (red) supporting a feasible transportation flow.

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Legend:

Gray edges: Original transportation network G with all possible routes and costs.

Red edges: A connected spanning subgraph $T \subseteq G$ minimizing transportation cost.

Theorem 2. Let $G = (V, E, w)$ be a directed graph with non-negative weights representing unit transportation costs. Define the regularized transportation cost function:

$$C_\epsilon(f) = \sum_{e \in E} \mathrm{f(e)} \cdot \mathrm{w\ (e)} + \varepsilon \sum_{e \in E} \mathrm{f(e)logf(e)} \quad (5)$$

where $\varepsilon > 0$, and f is a feasible flow over G. Then the problem of minimizing $C_\varepsilon(f)$ over the set of feasible flows has a unique solution.

Proof. The regularized cost function $C_\varepsilon(f)$ is the sum of a linear cost term and an entropic regularization term. The entropy term;

$$H(f) = \sum_{e \in E} \mathrm{f(e)} \cdot \log \mathrm{f\ (e)} \quad (6)$$

is strictly convex for f (e) > 0, while the linear term is convex. Therefore, the entire function Cε(f) is strictly convex on the interior of the feasible region defined by the flow conservation constraints and non-negativity of flows. Since the feasible set is convex and compact (due to bounded capacities or finiteness of supply and demand), the minimization problem admits a unique global minimizer.

Thus, there exists a unique optimal flow f ∗ that minimizes the entropically regularized cost.

Theorem 3. Let G = (V, E, w) be a directed graph with a feasible transportation flow f , and suppose that D ⊆ E is a dominating edge set in the sense that every vertex in V is incident to at least one edge in D. Then, there exists a feasible flow fD supported only on D such that the total transportation cost incurred by fD is no greater than that of f.

Proof. Let f be a feasible flow on the original edge set E. Since D is a dominating set, for every vertex involved in the transportation process, there exists at least one edge in D that can carry flow into or out of the vertex.

Construct a modified flow fD by re-routing the original flow f to pass only through the edges in D, using redistribution techniques that preserve flow conservation and demand-supply balance. This may involve decomposing f into flow paths and cycles and reassigning flows using only edges in D. Since the set D preserves connectivity and reachability (due to domination), the modified flow fD remains feasible. Additionally, because the re-routing avoids non- dominating edges often the cause of redundant or longer paths the overall cost of fD does not exceed that of f. In many practical cases, the cost is

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strictly reduced due to removal of inefficient or detour paths.

Hence, the support of the flow can be minimized without increasing total cost, provided edge domination is maintained.

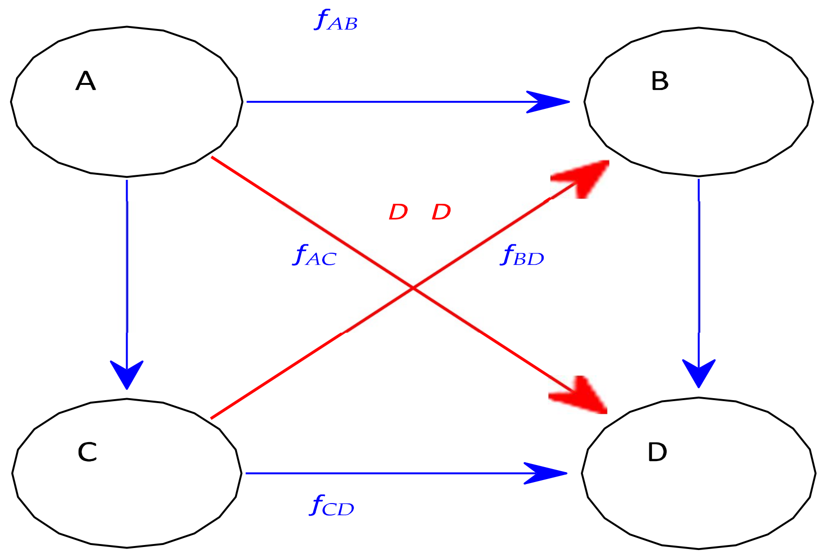

Figure 3: An illustration of a feasible flow f (blue) and a dominating edge set D (red) covering all vertices in the graph.

Legend:

Blue arrows: Edges used in the original feasible flow f.

Red arrows: Edges in the dominating set D.

Conclusion

This study has demonstrated the value of integrating graph-theoretic methods into the analysis and optimization of transportation logistics. By modeling transportation networks as graphs, complex logistics problems become more tractable and visually intuitive, enabling the application of powerful algorithms that identify cost-effective and reliable transportation strategies. Furthermore, incorporating regularization techniques enhances the stability, uniqueness, and robustness of the solutions, which is crucial for handling real-world uncertainties in demand, cost, and route availability. Beyond optimization, graph-theoretic approaches provide useful insights into the structural resilience and vulnerability of transport systems, guiding planners in infrastructure development and risk mitigation. Overall, this integration bridges

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theoretical rigor with practical logistics management, contributing meaningfully to both research and industry applications.

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